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# Solution of Linear Partial Integro-Differential Equations Using Mahgoub Transform

**Abstract**

In this paper, we used Mahgoub transform for solving linear partial integro-differential equations. The technique is described and illustrated with application. This technique gives the exact results using very less computational work.

**Keywords:** Linear Partial Integro-Differential Equation, Mahgoub Transform, Convolution Theorem, Inverse Mahgoub Transform.

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**Introduction**

Mathematical modeling of real life problems usually results in functional equations e.g. differential equations, partial differential equations, integral equations, integro-differential equations, stochastic equations, delay differential equations, partial integro-differential equations and others. In particular partial integro-differential equations arise in many scientific and engineering applications such as mathematical physics, visco-elasticity, finance, heat transfer, diffusion process, nuclear reactor dynamics, in general neutron diffusion, nano-hydrodynamics and fluid dynamics.

The general linear partial integro-differential equation is given by

$$\sum_{i=0}^{m} a_i \frac{\partial^i u(x,t)}{\partial t^i} + \sum_{i=0}^{n} b_i \frac{\partial^i u(x,t)}{\partial x^i} + cu + \sum_{i=0}^{r} d_i \int_0^t k_i\,(t,s) \frac{\partial^i u(x,s)}{\partial x^i} + f(x,t) = 0 \ldots\ldots (1)$$

(with prescribed conditions), where the kernels $k_i(t,s)$ and $f(x,t)$ are known functions and $a_i$, $b_i$, $c$ and $d_i$ are constants or functions of $x$.

The Mahgoub transform of the function $F(t)$is defined as [6]:

$$M\{F(t)\} = v\int_0^\infty F(t)e^{-vt}\,dt$$
$$= H(v), t \geq 0, k_1 \leq v \leq k_2$$

where $\mathrm{M}$ is Mahgoub transform operator.

The Mahgoub transform of the function $\mathrm{F(t)}$ for $\mathrm{t} \geq 0$ exist if $\mathrm{F(t)}$ is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Mahgoub transform of the function $\mathrm{F(t)}$.

**Review of Literature**

Appell et al. [1] discussed the partial integral operators and integro differential equations. Bahuguna and Dabas [2] gave existence and uniqueness of a solution to a PIDE by the method of lines. Yanik and Fairweather [3] used finite element methods for parabolic and hyperbolic partial integro-differential equations. Dehghan [4] discussed the solution of a partial integro-differential equation arising from viscoelasticity. Efficient solution of a partial integro-differential equation in finance was given by Sachs and Strauss [5]. Mahgoub [6] gave the new integral transform "Mahgoub Transform".

Mahgoub and Alshikh [7] applied Mahgoub transform for solving partial differential equations. Fadhil [8] gave the convolution for Kamal and Mahgoub transforms. Taha et. al. [9] gave the dualities between Kamal & Mahgoub integral transforms and some famous integral transforms. For modeling biofluids flow in fractured biomaterials, Zadeh [10] gave an integro-partial differential equation. Thorwe and Bhalekar [11] used Laplace transform method for solving partial integro-differential equations. Mohand and Tarig [12] applied Elzaki transform method for solving partial integro-differential equations. Aboodh et al. [13] gave the solution of partial integro-differential equations by using Aboodh and double Aboodh transform methods. Mohand [15] used double Elzaki transform method for solving partial integro-differential equations. Aggarwal et al. [16] discussed a new application of Mahgoub transform for solving linear Volterra integral equations. Aggarwal et al. [17] solved the linear Volterra integro-differential equations of second kind using Mahgoub transform.

The object of the present study is to determine exact solutions for linear partial integro-differential equations using Mahgoub transform without large computational work.

**Linearity Property of Mahgoub Transforms**

If $M\{F(t)\} = H(v)$ and $M\{G(t)\} = I(v)$ then

$$M\{aF(t) + bG(t)\} = aM\{F(t)\} + bM\{G(t)\}$$

$= aH(v) + bI(v)$,where $a, b$ are arbitrary constants.

**Mahgoub Transform of Some Elementary Functions [6, 8]**

| S.N. | $F(t)$ | $M\{F(t)\} = H(v)$ |
|---|---|---|
| 1. | $1$ | $1$ |
| 2. | $t$ | $\frac{1}{v}$ |
| 3. | $t^2$ | $\frac{2!}{v^2}$ |
| 4. | $t^n, n \in N$ | $\frac{n!}{v^n}$ |
| 5. | $e^{at}$ | $\frac{v}{v-a}$ |
| 6. | $sinat$ | $\frac{av}{v^2+a^2}$ |
| 7. | $cosat$ | $\frac{v^2}{v^2+a^2}$ |
| 8. | $sinhat$ | $\frac{av}{v^2-a^2}$ |
| 9. | $coshat$ | $\frac{v^2}{v^2-a^2}$ |

**Mahgoub transform of some partial derivatives of the function $u(x,t)$[7]**

If $M\{u(x,t)\} = H(x,v)$ then

(a) $M\left\{\frac{\partial u(x,t)}{\partial t}\right\} = vH(x,v) - vu(x,0) \ldots..(3)$

(b) $M\{\frac{\partial^2 u(x,t)}{\partial t^2}\} = v^2H(x,v) - v^2u(x,0)$
$-vu_t(x,0) \ldots \ldots.(4)$

(c) $M\left\{\frac{\partial^n u(x,t)}{\partial t^n}\right\} = v^nH(x,v) - v^nu(x,0)$
$-v^{n-1}u_t(x,0) - \cdots \ldots$
$- vu_{ttt\,....(n-1)times}(x,0) \ldots.(5)$

(d) $M\left\{\frac{\partial u(x,t)}{\partial x}\right\} = \frac{dH(x,v)}{dx} \ldots \ldots \ldots (6)$

(e) $M\left\{\frac{\partial^2 u(x,t)}{\partial x^2}\right\} = \frac{d^2H(x,v)}{dx^2} \ldots \ldots..(7)$

(f) $M\left\{\frac{\partial^n u(x,t)}{\partial x^n}\right\} = \frac{d^nH(x,v)}{dx^n} \ldots \ldots..(8)$

**Convolution of two functions [8]**

Convolution of two functions $F(t)$ and $G(t)$ is denoted by $F(t) * G(t)$ and it is defined by

$$F(t) * G(t) = F * G = \int_0^t F(x)G(t-x)dx$$
$$= \int_0^t F(t-x)G(x)dx$$

**Convolution theorem for Mahgoub transforms [8]**

If $M\{F(t)\} = H(v)$ and $K\{G(t)\} = I(v)$ then

$$M\{F(t) * G(t)\} = \frac{1}{v}M\{F(t)\}M\{G(t)\} = \frac{1}{v}H(v)I(v)$$

**Inverse Mahgoub Transform**

If $M\{F(t)\} = H(v)$ then $F(t)$ is called the inverse Mahgoub transform of $H(v)$ and mathematically it is defined as

$F(t) = M^{-1}\{H(v)\}$

where $M^{-1}$ is the inverse Mahgoub transform operator.

**Inverse Mahgoub transform of some elementary functions**

| S.N. | $H(v)$ | $F(t) = M^{-1}\{H(v)\}$ |
|---|---|---|
| 1. | $1$ | $1$ |
| 2. | $\frac{1}{v}$ | $t$ |
| 3. | $\frac{1}{v^2}$ | $\frac{t^2}{2!}$ |
| 4. | $\frac{1}{v^n}, n \in N$ | $\frac{t^n}{n!}$ |
| 5. | $\frac{v}{v-a}$ | $e^{at}$ |
| 6. | $\frac{v}{v^2+a^2}$ | $\frac{sinat}{a}$ |
| 7. | $\frac{v^2}{v^2+a^2}$ | $cosat$ |
| 8. | $\frac{v}{v^2-a^2}$ | $\frac{sinhat}{a}$ |
| 9. | $\frac{v^2}{v^2-a^2}$ | $coshat$ |

**Mahgoub transform for linear partial integro-differential equations**

In this section, we present Mahgoub transform for solving linear partial integro-differential equations given by (1). In this work, we will assume that the kernels $k_i(t,s)$ of (1) are difference kernel that can be expressed by difference $(t-s)$. The linear partial integro-differential equation (1) can thus be expressed as

$$\sum_{i=0}^{m} a_i \frac{\partial^i u(x,t)}{\partial t^i} + \sum_{i=0}^{n} b_i \frac{\partial^i u(x,t)}{\partial x^i} + cu$$
$$+ \sum_{i=0}^{r} d_i \int_0^t k_i\,(t-s) \frac{\partial^i u(x,s)}{\partial x^i}$$
$$+ f(x,t) = 0 \ldots \ldots.(9)$$

Applying the Mahgoub transform to both sides of(9), we have

$$\sum_{i=0}^{m} a_i M\left\{\frac{\partial^i u(x,t)}{\partial t^i}\right\} + \sum_{i=0}^{n} b_i M\left\{\frac{\partial^i u(x,t)}{\partial x^i}\right\} + cM\{u\} + \sum_{i=0}^{r} d_i\, M\left\{\int_0^t k_i\,(t-s)\frac{\partial^i u(x,s)}{\partial x^i}\right\} + M\{f(x,t)\} = 0 \ldots \ldots \ldots..(10)$$

Using convolution theorem of Mahgoub transform and equations (5) and (8) in equation(10), we have

$$\sum_{i=0}^{m} a_i \left[v^iH(x,v) - v^iu(x,0) - v^{i-1}u_t(x,0) - \cdots \ldots - vu_{ttt\,....(i-1)times}(x,0)\right]$$

$+\sum_{i=0}^{n} b_i \frac{d^i H(x,v)}{dx^i} + cH(x,v)$

$+\sum_{i=0}^{r} d_i \frac{1}{v} \bar{k}_i(v) \frac{d^i H(x,v)}{dx^i} + \bar{f}(x,v) = 0 \ldots\ldots(11)$

where $M\{u(x,t)\} = H(x,v), M\{k_i(t)\} = \bar{k_i}(v)$ and $M\{f(x,t)\} = \bar{f}(x,v)$.

After using prescribed conditions, equation (11) represents an ordinary differential equation with dependent variable $H(x,v)$. After solving this ordinary differential equation and taking inverse Mahgoub transform of $H(x,v)$ , we have the required solution $u(x,t)$ of equation (1).

**Applications**

In this section, an application is given in order to demonstrate the effectiveness of Mahgoub transform for solving linear partial integro-differential equation.

Consider the linear partial integro-differential equation [11-13]

$u_{tt} = u_x + 2\int_0^t (t-s)\, u(x,s) ds - 2e^x \ldots (12)$

with initial conditions

$u(x,0) = e^x, u_t(x,0) = 0 \ldots\ldots (13)$

and boundary condition

$u(0,t) = cost \ldots\ldots\ldots\ldots\ldots\ldots. (14)$

Applying Mahgoub transform to both sides of equation (12), we have

$$M\{u_{tt}\} = M\{u_x\} + 2M\left\{\int_0^t (t-s)\, u(x,s) ds\right\} - 2e^x M\{1\} \ldots\ldots\ldots\ldots (15)$$

Using convolution theorem of Mahgoub transform and equations (4) and (6) in equation (15), we have

$v^2 H(x,v) - v^2 u(x,0) - v u_t(x,0)$

$= \frac{dH(x,v)}{dx} + \frac{2}{v^2} H(x,v) - 2e^x \ldots\ldots\ldots (16)$

Now using equation(13) in equation (16), we have

$$\frac{dH(x,v)}{dx} + \left(\frac{2}{v^2} - v^2\right) H(x,v) = (2 - v^2)e^x \ldots\ldots. (17)$$

which is an ordinary linear differential equation.

The general solution of equation (17) is give by

$$H(x,v) = e^x \left(\frac{v^2}{1+v^2}\right) + ce^{-\left(\frac{2}{v^2} - v^2\right)x} \ldots. (18)$$

Now, using equation (14), we have

$M\{u(0,t)\} = H(0,v) = M\{cost\} = \frac{v^2}{1+v^2} \ldots (19)$

Using equation (19) and equation(18), we have

$c = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$

Substituting the value of $c$ from equation (20) into equation (18), we have

$H(x,v) = e^x \left(\frac{v^2}{1+v^2}\right) \ldots\ldots\ldots\ldots. (21)$

Operating inverse Mahgoub transform on both sides of equation (21), we have

$$u(x,t) = M^{-1}\{H(x,v)\} = e^x M^{-1}\left\{\frac{v^2}{1+v^2}\right\} = e^x cost \ldots\ldots\ldots\ldots. (22)$$

which is the required exact solution of equation (12) with equations (13) and (*14*).

**Conclusion**

In this paper, we have successfully developed the Mahgoub transform for solving linear partial integro-differential equation. The given application shows that the exact solution have been obtained using very less computational work and spending a very little time. The proposed scheme can be applied for other linear partial integro-differential equations.

**Endnotes**


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